Nonlinear Schrödinger soliton in a time-dependent quadratic potential

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We examine the one-soliton solution of the nonlinear Schrödinger equation (NLSE) with an external potential of the form of $V(x,t)=f_1(t)x+f_2(t)x^2$ where $f_1(t)$ and $f_2(t)$ are arbitrary functions of t except that $f_2(t)$ stays above a certain negative value. It is shown that, while the center of the soliton obeys Newton's equation with the potential $V(x,t)$, the internal structure of the soliton is determined by the NLSE of the "body-fixed" coordinate system. The soliton structure is found to be independent of $f_1(t)$. The soliton is rigid if $f_2(t)$ is t independent but it can be diffused when $f_2(t)$ varies rapidly. Numerical experiments, however, show that the soliton withstands very rapid variations of $f_2(t)$.

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I. INTRODUCTION

Let us consider the nonlinear Schrödinger equation (NLSE) in 1+1 dimensions with an external potential

$$i\psi_t=-\frac{1}{2m}\psi_{xx}-g|\psi|^2\psi+V(x,t)\psi\ , \tag{1.1}$$

where $\psi=\psi(x,t)$, $\psi_t=\partial\psi/\partial t$, $\psi_{xx}=\partial^2\psi/\partial x^2$, $m\,(>0)$ is the "mass," $g\,(>0)$ is a constant, and $V(x,t)$ is the external potential. It is understood, in this paper, that ψ is normalized as $\int_{-\infty}^{\infty}|\psi(x,t)|^2dx=1$. The NLSE finds applications in many areas of physics. It can be regarded as an equation of the Hartree-Fock type (with units such that $\hbar=1$) for a many-body system in quantum mechanics [1]. Then the nonlinear Schrödinger soliton described by Eq. (1.1) simulates a many-body bound system (like an atomic nucleus) placed in an external potential field. The NLSE also describes electron (Langmuir) waves in plasma physics, propagation of optical pulses in a nonlinear optical fiber, and so on [2,3]. The purpose of this paper is to examine the behavior and structure of the soliton of Eq. (1.1) when the external potential is of the quadratic form

$$V(x,t)=V_1(x,t)+V_2(x,t)\ ,\quad V_n(x,t)\equiv f_n(t)x^n\ , \tag{1.2}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions of t except that $f_2(t)$ satisfies certain conditions which we will discuss. What we find will give insight into cases of more general form of the external potential.

Analysis of Eq. (1.1) is facilitated by transforming the coordinate from the "laboratory" system to the "body-fixed" system. The latter system is such that the center of the soliton is at rest at the origin. For $V(x,t)$ of Eq. (1.2) this transformation completely separates the center-of-mass motion and the internal structure of the soliton. The transformed equation determines the structure of the soliton which turns out to be independent of its motion. This method is an application of Husimi's method [4] for solving the usual linear Schrödinger equation with the potential of Eq. (1.2). It also comprises the ingenious method used by Chen and Liu [5,6], who examined the NLSE with time-independent linear and quadratic external potentials.

In Sec. II we present the method and discuss general aspects of the soliton solution. We examine explicit examples in Sec. III. We are particularly interested in the stability of the soliton when $f_2(t)$ becomes negative, i.e., $V_2(x,t)$ becomes an *inverted* harmonic oscillator (HO) potential. We also discuss the stability of the soliton against rapid time dependence of the external potential. The stability of the soliton in an inverted HO potential will become relevant in developing approximation methods for $V(x,t)$ of more general form such that $\partial^2V(x,t)/\partial x^2<0$ in some spatial region. A summary is given in Sec. IV.

II. THE METHOD

In the absence of the external potential, $V(x,t)=0$, Eq. (1.1) has the well-known one-soliton solution given by

$$\psi(x,t)=A_0(x-vt)e^{i[mvx-(\epsilon_0+mv^2/2)t]}\ , \tag{2.1}$$

$$A_0(x)=\left[\frac{\kappa}{2}\right]^{1/2}\operatorname{sech}(\kappa x)\ , \tag{2.2}$$

$$\epsilon_0=-\frac{\kappa^2}{2m}\ ,\quad \kappa=\tfrac{1}{2}mg\ . \tag{2.3}$$

The $A_0(x)$, which determines the shape of the "free" soliton, is the bound-state solution of the t-independent equation

$$-\frac{1}{2m}A_{0xx}-gA_0^3=\epsilon_0A_0\ . \tag{2.4}$$

For the external $V(x)=m\alpha x$ where α is a constant, Chen and Liu [5] showed that Eq. (1.1) is integrable. Their one-soliton solution shows that the soliton behaves exactly like a classical particle of mass m subject to the potential $V(x)=m\alpha x$. The shape of the soliton remains the same as the free soliton given by A_0 of Eq. (2.2). Chen and Liu [6] further pointed out that Eq. (1.1) with the HO potential $V(x)=\frac{1}{2}m\omega^2x^2$ has a one-soliton solution which again behaves like a classical particle. The structure of the soliton in this case becomes different from the free soliton, but Chen and Liu did not examine this aspect of the problem in any detail.

Chen and Liu [5,6] employed a transformation in examining the NLSE with an external potential. Their transformation belongs to the type that was proposed long ago by Husimi [4] in order to solve the usual linear Schrödinger equation for the generalized HO potential $V(x,t)$ of Eq. (1.2). Husimi's transformation is to change the space variable from x to x' where

$$x'=x-\xi(t) \ . \tag{2.5}$$

This x' is the coordinate with respect to the moving origin $\xi(t)$. Later $\xi(t)$ will be taken as the center of mass of the soliton. Let us also write $\psi(x,t)$ as

$$\psi(x,t)=\phi(x',t)e^{im\dot{\xi}x'} \ , \tag{2.6}$$

where $\dot{\xi}=d\xi(t)/dt$. Substituting Eqs. (2.5) and (2.6) into Eq. (1.1) yields

$$i\phi_t=-\frac{1}{2m}\phi_{x'x'}-g|\phi|^2\phi+V_2(x',t)\phi + [m\ddot{\xi}+V_\xi(\xi,t)]x'\phi-[\tfrac{1}{2}m\dot{\xi}^2-V(\xi,t)]\phi \ , \tag{2.7}$$

where $V_\xi(\xi,t)=\partial V(\xi,t)/\partial\xi$. In deriving this we have used

$$V(x'+\xi,t)=V_2(x',t)+V(\xi,t)+x'V_\xi(\xi,t) \ , \tag{2.8}$$

which is valid for $V(x,t)$ of Eq. (1.2). We now require that $\xi(t)$ satisfy Newton's equation

$$m\ddot{\xi}=-V_\xi(\xi,t) \ . \tag{2.9}$$

Then we arrive at

$$\psi(x,t)=\chi(x',t)\exp\left[im\dot{\xi}x'+i\int^t L(t')dt'\right] \ , \tag{2.10}$$

$$L(t)=\tfrac{1}{2}m\dot{\xi}^2-V(\xi,t) \ , \tag{2.11}$$

$$i\chi_t=-\frac{1}{2m}\chi_{x'x'}-g|\chi|^2\chi+V_2(x',t)\chi \ . \tag{2.12}$$

Equation (2.12) looks like the original Eq. (1.1), but there are two crucial differences. (i) Equation (1.1) is for the laboratory system, whereas Eq. (2.12) is for the moving system with its origin at $x=\xi(t)$. (ii) The term V_1 is absent in Eq. (2.12) and parity with respect to x' is a good quantum number. This second feature allows Eq. (2.12) to have a bound-state solution such that $\int_{-\infty}^{\infty}|\chi(x',t)|^2x'dx'=0$. Here, by a "bound state" we mean a state confined in a finite spatial region; the density $|\chi(x',t)|^2$ does not have to be t independent [7]. Equation (2.12) determines the structure of the bound state, which is independent of $\xi(t)$. The center of mass of the bound state is at $x'=0$, i.e., $x=\xi(t)$. The x'-coordinate system is the "body-fixed system" of the bound state.

If $g=0$, Eq. (2.12) is the Schrödinger equation for a HO with a t-dependent frequency. Including Husimi's paper [4] there is vast literature on this subject [8]. Equation (2.12) with $g=0$ and $f_2(t)>0$ obviously has bound-state solutions. All of these bound states remain when the attractive nonlinear term is added. The lowest state is strongly affected by the nonlinear interaction and forms a soliton. It becomes more deeply bound and strongly localized while higher states are much less affected by the nonlinear interaction. If $f_2(t)<0$, $V_2(x,t)$ is an inverted HO potential; if $g=0$, Eq. (2.12) admits no bound state. If $g>0$, however, there can be a bound soliton state even when $f_2(t)<0$ as we will elucidate later.

Next let us examine the compatibility between Eqs. (1.1) and (2.9). This has to do with Ehrenfest's theorem of quantum mechanics [9]. Following the standard steps in deriving Ehrenfest's theorem we obtain

$$m\ddot{\xi}=\int_{-\infty}^{\infty}(g\rho_x-V_x)\rho dx \ , \tag{2.13}$$

where $\rho(x,t)=|\psi(x,t)|^2$. The term with $\rho_x=\partial\rho/\partial x$ on the right-hand side vanishes because $\int_{-\infty}^{\infty}\rho_x\rho dx=\frac{1}{2}[\rho^2]_{-\infty}^{\infty}=0$. Thus Ehrenfest's theorem in the usual form follows. For $V(x,t)$ of Eq. (1.2)

$$\int_{-\infty}^{\infty}V_x\rho dx=V_\xi(\xi,t) \tag{2.14}$$

obviously holds, and Eq. (2.13) is reduced to Eq. (2.9). This concludes the formal presentation of the method.

In order to be able to see some interesting features of the method without undue complication, let us first focus on the case in which $f_2(t)$ is t independent. Then χ of Eq. (2.12) can be written as $\chi(x',t)=A(x')e^{-i\epsilon t}$, where $A(x')$ is real. For the soliton we take the ground-state solution of Eq. (2.12). The solution for this case can be summarized as follows:

$$\psi(x,t)=A[x-\xi(t)]e^{iS(x,t)} \ , \tag{2.15}$$

$$S(x,t)=m\dot{\xi}(x-\xi)-\epsilon t+\int^t L(t')dt' \ , \tag{2.16}$$

$$-\frac{1}{2m}A_{xx}(x)-gA^3(x)+V_2(x)A(x)=\epsilon A(x) \ . \tag{2.17}$$

In Eq. (2.17) we have put $\xi=0$ so that $x'=x$ without losing generality. The soliton obtained in this way moves in the x space exactly like a classical particle of mass m [10]. No "radiation" takes place, that is, the energy of the ψ field remains contained in the soliton. The shape of the soliton, determined by $A(x')=A(x-\xi)$ of Eq. (2.17), remains the same throughout the course of motion. The solution is valid no matter how rapidly $f_1(t)$ varies with time [11,12].

III. EXAMPLES AND DISCUSSIONS

Let us consider four explicit examples for $V(x,t)$. All the quantities can be taken as dimensionless. In numerical illustrations for the examples we put $m=1$, $g=1$, and $\kappa=\frac{1}{2}$ throughout.

Example I: Linear potential with

$$V(x,t)=V_1(x,t)=m\alpha(t)x \ , \tag{3.1}$$

where $\alpha(t)$ is an arbitrary function of t. Since $V_2=0$, we find that $A(x)=A_0(x)$ and $\epsilon=\epsilon_0$. The $\xi(t)$ is determined by $m\ddot{\xi}=-\alpha(t)$. With this $\xi(t)$, $S(x,t)$ of Eq. (2.16) can be worked out. This is a straightforward generalization of Chen and Liu's solution for t independent α [5].

Example II; HO model with

$$V(x)=V_2(x)=\tfrac{1}{2}m\omega^2x^2 , \tag{3.2}$$

where ω is a constant. Equation (2.17) in this case becomes

$$-\frac{1}{2m}A_{xx}-gA^3+(\tfrac{1}{2}m\omega^2x^2)A=\epsilon A . \tag{3.3}$$

The relative strength of the external potential versus the nonlinear self-interaction can be gauged by $(m\omega)^{1/2}/\kappa$ where $\kappa=\frac{1}{2}mg$ as before. If $(m\omega)^{1/2}/\kappa\simeq 1$, the contribution of the two interactions to the soliton binding are about equal. However, such a case is uninteresting because the HO potential is so narrow that the soliton has no room to move around. We are interested in cases of $(m\omega)^{1/2}/\kappa \ll 1$.

We have not been able to solve Eq. (3.3) analytically; so we solved it numerically. For the strength parameter of the HO potential we took $\frac{1}{2}m\omega^2=0.0005$, which means $(m\omega)^{1/2}/\kappa\simeq 0.36$. Figures 1 and 2 show the effective potential

$$V_{\text{eff}}(x)=V(x)-gA(x)^2 , \tag{3.4}$$

for the ground and the first excited states, respectively. The ϵ for these states are -0.1266 and -0.018, respectively. These can be compared with $\epsilon_0=-0.125$ and $\omega/2=0.032$. Apart from the energy as a classical particle of mass m moving in the potential $V(\xi,t)$, the energy of the system is given by

$$\mathcal{E}=\int_{-\infty}^{\infty}\left[\frac{1}{2m}(A_x)^2-\tfrac{1}{2}gA^4+(\tfrac{1}{2}m\omega^2x^2)A^2\right]dx . \tag{3.5}$$

This $\mathcal{E}$ is not equal to ϵ. [If we replace the term $-gA^4/2$ with $-gA^4$ in Eq. (3.5), then $\mathcal{E}$ becomes equal to ϵ.] The $\mathcal{E}$ for the ground and excited states obtained above are -0.0401 and 0.0169, respectively.

The A of the ground state is not very different from A_0 of Eq. (2.2), while the A of the first excited state is like the first excited state (of odd parity) for the HO potential in the absence of the nonlinear term. We take the ground state for the soliton. The excited states are much more diffuse than the soliton state. The phase $S(x,t)$ of Eq. (2.16) can be explicitly worked out by using, for example, $\xi(t)=\xi_0\cos\omega t$. This S, combined with the approximation $A\simeq A_0$, gives Chen and Liu's solution [6].

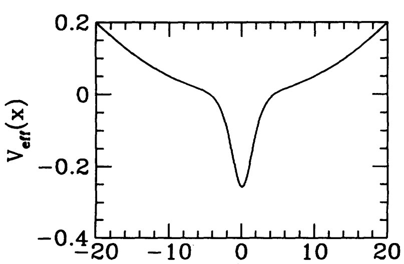

FIG. 1. The effective potential $V_{\text{eff}}(x)$ of Eq. (3.4) for the ground state of example II. The parameters of the model are $m=1$, $g=1$, and $\frac{1}{2}m\omega^2=0.0005$. The x, t, and $V(x)$ can be taken as dimensionless quantities.

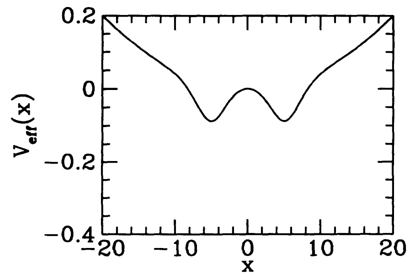

FIG. 2. The same as for Fig. 1, but for the first excited state.

Example III: Inverted HO model with

$$V(x)=V_2(x)=-\tfrac{1}{2}m\omega^2x^2 . \tag{3.6}$$

There is no particular difficulty regarding the phase function $S(x,t)$ of this model. Therefore, let us focus on the amplitude function $A(x)$. Equation (2.17) with $\xi=0$ reads

$$-\frac{1}{2m}A_{xx}-gA^3-(\tfrac{1}{2}m\omega^2x^2)A=\epsilon A . \tag{3.7}$$

If the nonlinear term $-gA^3$ is absent, there is no bound state. With $-gA^3$, however, Eq. (3.7) can have a bound-state solution. Imagine a trial function A localized around the origin. It produces an effective attractive potential $-gA^2$. If we delocalize A, $-gA^2$ becomes less attractive and the energy of the system increases. If A is further delocalized away from the origin, the energy begins to decrease because A feels more of $V(x)$ which is negative. Suppose a parameter λ measures the degree of localization of A. Then the energy as a function of λ will have a local minimum. This implies the existence of a bound state.

Let us put the above argument on a firmer ground. Note that Eq. (3.7) is equivalent to the variational problem,

$$\frac{\delta}{\delta A}[\mathcal{E}(A)-\epsilon A^2]=0 , \tag{3.8}$$

where $\mathcal{E}(A)$ is the energy functional defined by Eq. (3.5) with the sign of the HO potential reversed. As an illustration, assume that

$$A(x,\lambda)=\left[\frac{\lambda}{2}\right]^{1/2}\operatorname{sech}(\lambda x) . \tag{3.9}$$

Then we obtain $\mathcal{E}(A)$ as a function of λ,

$$\mathcal{E}(\lambda)=\frac{1}{6m}\left[\lambda^2-mg\lambda-\frac{\pi^2}{4}\left[\frac{m\omega}{\lambda}\right]^2\right] . \tag{3.10}$$

The $d\mathcal{E}(\lambda)/d\lambda=0$ has two real positive roots if

$$\tfrac{1}{2}m\omega^2 < (\tfrac{3}{4})^3 \frac{1}{2\pi^2}\frac{\kappa^4}{m} , \qquad (3.11)$$

where $\kappa = \frac{1}{2}mg$. The larger one of the two roots corresponds to the minimum of $\mathcal{E}(\lambda)$. Condition (3.11) can also be written as $(m\omega)^{1/2}/\kappa < (\frac{3}{4})^{3/4}\pi^{-1/2} \simeq 0.70$.

We have numerically solved Eq. (3.7) and found a bound state when $\frac{1}{2}m\omega^2 \lesssim 0.0007$, i.e., $(m\omega)^{1/2}/\kappa \lesssim 0.39$. Equation (3.11) leads to $\frac{1}{2}m\omega^2 < 0.0013$, which is too lenient. Figure 3 shows the effective potential $V_{\text{eff}}(x)$ for the bound state obtained for $\frac{1}{2}m\omega^2 = 0.0005$. We found $\epsilon = -0.1230$ and $\mathcal{E} = -0.0434$ for this state. If $\frac{1}{2}m\omega^2$ exceeds 0.0007, the bound state begins to be unstable. In order to see this we solved Eq. (1.1) starting with $\psi(x,t)$ of Eq. (2.1) with $v=0$. The ψ quickly settles down to a stationary state if $\frac{1}{2}m\omega^2 \leq 0.0007$. Figure 4 shows how the bound state "decays" when $\frac{1}{2}m\omega^2 = 0.001$. In this case, however, we can restore the soliton, i.e., pull the diffused soliton back to a compact form, by reducing $\frac{1}{2}m\omega^2$ to $\lesssim 0.007$ before the soliton gets too diffused.

Example IV: Forced HO model with

$$V(x,t) = \tfrac{1}{2}m\omega^2 x^2 - m\omega^2 F(t)x , \qquad (3.12)$$

where $m\omega^2 F(t)x$ is the additional perturbation. By solving Newton's equation for $\xi(t)$ we find

$$\xi(t) = \omega \int_{-\infty}^{t} F(t')\sin[\omega(t-t')]dt' . \qquad (3.13)$$

A term like $\xi_0 \cos\omega t$ can be added to the right-hand side of Eq. (3.13). The amplitude function $A(x-\xi)$ remains the same as that of example II. The phase $S(x,t)$ can be worked out by substituting Eq. (3.13) into Eq. (2.16). Let us emphasize that $F(t)$ can be chosen arbitrarily as long as the integral of Eq. (3.13) can be defined. No matter how violently $F(t)$ varies, the solution remains valid. This aspect is essentially the same as what was recently emphasized for the linear Schrödinger equation for the forced HO [13]. With this note we end the explicit examples.

Let us now consider the case of t dependent $f_2(t)$. If the variation of $f_2(t)$ is sufficiently gentle, the ψ of Eq. (2.15) can be taken as an adiabatic approximation. In that case ϵ becomes t dependent, and ϵt in Eq. (2.16) has to be replaced by $\int^t \epsilon(t')dt'$. If the variation of $f_2(t)$ becomes wild, the $\chi(x,t)$ which was in the ground state of Eq. (2.17) can get an admixture of higher states and become diffuse. If $f_2(t)<0$, the $\chi(x,t)$ may escape to infinity. Below Eq (1.2) we stated that "$f_2(t)$ can be negative provided that $f_2(t)$ satisfies certain conditions." The conditions we meant are that $f_2(t)$ varies gently and it stays above the negative value which we estimated in example III.

We have done a variety of numerical experiments on Eq. (1.1). We already emphasized that the soliton survives any rapid variation of $f_1(t)$. This was amply confirmed by our numerical solutions. We are more interested in situations in which $f_2(t)$ varies rapidly. We let $f_2(t)$ vary within the range of $-0.0007 < f_2(t) \lesssim 0.001$. Note that $f_2(t) = 0.001$ corresponds to $(m\omega)^{1/2}/\kappa = 0.42$. This is indeed a very strong HO potential. We were surprised to find that the soliton does not break even when $f_2(t)$ changes suddenly. We found no significant trace of "radiation" emitted from the soliton. Figure 5 shows that the soliton easily survives the sudden and large change of $f_2(t)$ from 0.001 to -0.0005. The condition of gentle variation of $f_2(t)$ stated in the preceding paragraph is sufficient for the soliton survival but it is certainly not necessary.

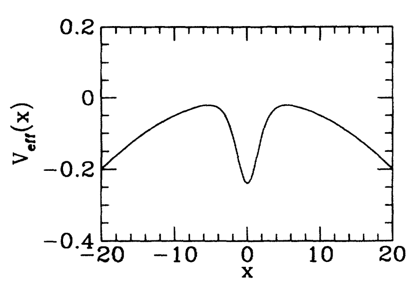

FIG. 3. The same as for Fig. 1, but for the inverted HO potential of example III.

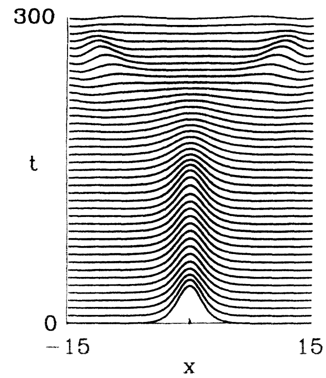

FIG. 4. The variation of the density $\rho(x,t)$ showing how the soliton decays in the inverted HO potential of example III: $\frac{1}{2}m\omega^2 = 0.001$. The x, t, and $\rho(x,t)$ can be taken as dimensionless quantities.

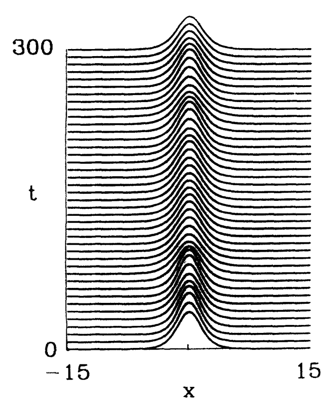

FIG. 5. The variation of the density $\rho(x,t)$ showing how the soliton survives a sudden change of $f_2(t)$, from $f_2(t<60)=0.001$ to $f_2(t>60)=-0.0005$. The x, t, and $\rho(x,t)$ can be taken as dimensionless quantities.

The $V(x,t)$ of Eq. (1.2) has a constant curvature as a function of x. If the curvature varies with x, our method does not apply exactly. However, if the variation of the curvature is very small over the width of the soliton ($\simeq 1/\kappa$), the method will be very effective. When the soliton is in a region where the curvature of $V(x,t)$ is negative, i.e., $\partial^2 V(x,t)/\partial x^2 < 0$, the soliton feels something like an inverted HO potential. We suspect that the soliton can easily survive such an environment. The restriction on the curvature found for example III can probably be much relaxed if $V(x,t)$ is bounded from below. As an extreme example, consider the repulsive δ-function potential $V(x)=G\delta(x)$. Equation (2.17) with V_2 replaced by $G\delta(x)$ is

$$-\frac{1}{2m}A_{xx}-gA^3+G\delta(x)A=\epsilon A \ . \tag{3.14}$$

This equation has a bound-state solution with

$$\epsilon=-\frac{\kappa'^2}{2m} \ , \quad \kappa'=\tfrac{1}{2}m(g-2G) \ , \tag{3.15}$$

which is valid if $2G<g$ [14]. Imagine that the δ function is simulated by something like a Gaussian function with a very small width. Then the curvature of this potential at $x=0$ will be very large in magnitude [15].

IV. SUMMARY

We have shown for the soliton described by the NLSE (1.1) with the t-dependent quadratic potential of Eq. (1.2) that its internal structure can be separated from the center-of-mass motion of the soliton. Equation (1.1) is reduced to Newton's equation (2.9) for the center of mass and Eq. (2.12) for the structure of the soliton. The soliton structure is independent of the linear term $V_1(x,t)$; hence the soliton withstands any rapid variation of $f_1(t)$. The soliton can be diffused, in principle, if $f_2(t)$ varies rapidly. We found through numerical experiments, however, that the soliton is extremely tenacious against rapid variations of $f_2(t)$. We examined in detail the case of $V(x,t)$ being an inverted harmonic-oscillator potential, i.e., $f_2(t)<0$. The soliton can survive such a situation unless $f_2(t)$ is very large in magnitude and remains so for a long time.

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[2] For the various applications of the NLSE, see, e.g., Y. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989), and references cited therein.

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[7] Consider, for example, the case such that V_2 is t independent and positive. Equation (2.12) has an infinite number of stationary bound states. In addition, there will be nonstationary states in which a wave packet oscillates. For these nonstationary states, which are like the coherent states for the pure HO potential, the density $|\chi(x',t)|^2$ is t dependent and $\int_{-\infty}^{\infty}|\chi(x',t)|^2 x'dx' \neq 0$. We are interested in the stationary bound states rather than those like the coherent states. When V_2 becomes t dependent, the density $|\chi(y,t)|^2$ becomes t dependent even for the "stationary" bound states. This is what we meant by saying that $|\chi(y,t)|^2$ does not have to be t independent.

[8] We quote just two papers which we found relatively more comprehensive and relevant; H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. **10**, 1448 (1969); I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Phys. Rev. D **2**, 1371 (1970).

[9] L. I. Schiff, *Quantum Mechanics,* 3rd ed. (McGraw-Hill, New York, 1968), Chap. 2.

[10] There is a subtle departure from the classical particle. The ψ remembers the history of motion through its phase factor $e^{iS(x,t)}$. When two solitons collide, the phase difference between them may have nontrivial effects.

[11] As noted in Ref. [3], the meanings of x and t are reversed in fiber-optics applications. A potential rapidly varying with time is interpreted as a rapid spatial variation of the index of refraction. The feature that a soliton can withstand a potential rapidly varying with time in the present paper would imply that an optical soliton can withstand a rapid spatial variation in the index of refraction.

[12] We have assumed the simplest nonlinearity, i.e., the cubic term $-g|\psi|^2\psi=-g\rho\psi$. The method we have developed can handle more complicated nonlinearities like $\{\int F[\rho(x',t)]v(x-x')dx'\}\psi(x,t)$.

[13] Y. Nogami, Am. J. Phys. **59**, 64 (1991).

[14] This bound-state problem can be solved in a way similar to that for $-(1/2m)A_{xx}+gA^3-G\delta(x)A=\epsilon A$, which was discussed by Y. Nogami, M. Vallières, and W. van Dijk, Am. J. Phys. **44**, 886 (1976), and L. L. Foldy, Am. J. Phys. **44**, 1193 (1976).

[15] We examined Eq. (1.1) with t independent $V(x)$ in the form of a step function and also a repulsive barrier. We found that the soliton impinging on a potential step or barrier is virtually unbreakable: Y. Nogami and F. M. Toyama, Phys. Lett. A **184**, 245 (1994).